Statistical mechanics of nonrelativistic charged particles in a constant magnetic field

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The statistical mechanics of a system of nonrelativistic charged particles in a constant magnetic field is discussed. The spatial dimension D is arbitrary, with $D \geqslant 3$ assumed. Calculations are presented from first principles using the effective action method. For $D \geqslant 5$ the system has a phase transition with a Bose condensate. We show how the effective action method deals in a very natural way with the condensate, and study its role in the magnetization of the gas. For large values of the magnetic field we show how the magnetized gas in D spatial dimensions behaves like the free Bose gas in $D-2$ spatial dimensions. Even though for $D=3$ the magnetized gas does not have a phase transition for any nonzero value of the magnetic field, we show how the specific heat starts to resemble the result for the free gas as the magnetic field is reduced. A number of analytical approximations for the magnetization and specific heat are given, and compared with numerical results. In this way we are able to study in precise detail how the $B \to 0$ limit of the magnetized gas is achieved. [S1063-651X(99)08211-2]

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I. INTRODUCTION

One system which has received considerable attention consists of charged spinless bosons in a homogeneous magnetic field. Initial work on the nonrelativistic case [1] was substantially improved on by Schafroth [2], and was then generalized to dimensions other than three by May [3,4]. More recently, several detailed studies have been made of an ideal system of nonrelativistic charged bosons, both numerically [5] and from a theoretical point of view in both two [6] and three [7–10] dimensions. Also, a detailed study of the boson gas in three dimensions with trapping harmonic potentials and the presence of a magnetic field has recently been made using the path-integral formalism [11]. The behavior of the relativistic case has also been discussed [9,12–14].

The outline of our paper is as follows. In Sec. II we give a brief description of the free Bose gas in D spatial dimensions. The purpose of this is to allow us to later compare the analogous results for the magnetized gas to these free field ones. Section III presents the effective action method, and and applies it to the magnetized gas. We concentrate initially on the specific heat, and show how the presence of the magnetic field alters the behavior from that found for the free gas. We study numerically what happens for large and small magnetic fields. The reason for concentrating on the specific heat is that the specific heat maximum can be used as a signature for Bose-Einstein condensation in cases where there is no phase transition. This was used by Pathria [15] in finite volume systems, and more recently for confined Bose gases [16,17]. Results are also obtained for the magnetization. In Sec. IV we describe how it is possible to obtain analytical results for the critical temperature (when $D \geqslant 5$), the magnetization, and the specific heat when the magnetic field is weak. We are able to analytically confirm the numerical results concerning the limit $B \to 0$. For the $D=3$ gas we give approximations valid at the critical temperature for free Bose gas. (Previous approximations were only valid for temperatures larger than this.)

II. BOSE-EINSTEIN CONDENSATION WITH NO EXTERNAL FIELD

In this section we wish to review very briefly some of the basic properties of the free Bose gas. The spatial dimension D will be arbitrary. Although there may be no direct physical relevance to cases with $D>3$, it is still instructive to study these cases because the nature of the phases transition differs from the $D=3$ situation. It also allows us to make contact with previous work for $D \neq 3$. For $D=3$ the analysis is standard textbook material [18–21]. The absence of Bose-Einstein condensation when $D=2$ is also widely known [22,3]. Spatial dimensions $D>3$ have also been studied [23,24].

We will consider an ideal gas of N spinless bosons confined in a large box of volume V in D spatial dimensions. The infinite volume limit will be taken with $\mathcal{N}=N/V$ fixed, as is conventionally done. The energy levels of the system are

$$E_{n_i} = \frac{1}{2m} \sum_{i=1}^{D} \left(\frac{2\pi n_i}{L_i} \right)^2 \tag{2.1}$$

before the infinite volume limit is taken if we impose periodic boundary conditions on the field. We set $\hbar = 1$ throughout the paper. In Eq. (2.1), $n_i = 0, \pm 1, \pm 2, \ldots,$ and L_i is the length of the box in the ith direction. In Sec. III we will show how the effective action formalism can be used to study this problem (see Refs. [8,25,26] for reviews). For now we stick with the conventional thermodynamic expressions.

The internal energy is given by

$$U = \sum_{n_i} \frac{E_{n_i}}{(e^{\beta(E_{n_i}-\mu)}-1)}, \tag{2.2}$$

where μ is the chemical potential, and $\beta = T^{-1}$ in units with the Boltzmann constant $k_B = 1$. The grand canonical ensemble is used here.

By taking the infinite volume limit, we can replace the sums in Eq. (2.2) over n_i with integrals. After expanding the denominator of Eq. (2.2) in a geometric series, and using Eq. (2.1) for the energy levels, the integrals over n_i may be performed with the result

$$U=\frac{D}{2}\beta V\left(\frac{m}{2\pi\beta}\right)^{D/2}\mathrm{Li}_{(D+2)/2}[e^{\beta\mu}]. \tag{2.3}$$

We have defined the polylogarithm function $\mathrm{Li}_p[z]$ by

$$\mathrm{Li}_p[z]=\sum_{k=1}^{\infty}\frac{z^k}{k^p}. \tag{2.4}$$

We note the property (assuming $p>1$)

$$\mathrm{Li}_p[1]=\zeta_R(p), \tag{2.5}$$

where $\zeta_R(p)$ denotes the Riemann ζ function.

The essential feature of Bose-Einstein condensation as a phase transition is exhibited in the behavior of the specific heat. The specific heat at constant volume C_v is defined by

$$C_v=\left(\frac{\partial U}{\partial T}\right)_{V,N}=-\beta^2\left(\frac{\partial U}{\partial\beta}\right)_{V,N}. \tag{2.6}$$

The particle number N, given by

$$N=V\left(\frac{m}{2\pi\beta}\right)^{D/2}\mathrm{Li}_{D/2}[e^{\beta\mu}], \tag{2.7}$$

is held fixed when computing the derivative in Eq. (2.6). The chemical potential is not fixed. From Eqs. (2.3) and (2.6) we find

$$C_v=V\left(\frac{m}{2\pi\beta}\right)^{D/2}\left[\frac{D(D+2)}{4}\mathrm{Li}_{(D+2)/2}[e^{\beta\mu}]-\frac{D}{2}\beta(\beta\mu)'\mathrm{Li}_{D/2}[e^{\beta\mu}]\right]. \tag{2.8}$$

Here $'$ denotes the derivative with respect to β, holding V and N fixed. To calculate $(\beta\mu)'$ we differentiate both sides of Eq. (2.7) to give

$$(\beta\mu)'=\frac{D}{2\beta}\frac{\mathrm{Li}_{D/2}[e^{\beta\mu}]}{\mathrm{Li}(D-2)/2[e^{\beta\mu}]}. \tag{2.9}$$

Substitution of Eq. (2.9) into Eq. (2.8), and use of Eq. (2.7), shows that

$$\frac{C_v}{N}=\frac{D(D+2)}{4}\frac{\mathrm{Li}_{(D+2)/2}[e^{\beta\mu}]}{\mathrm{Li}_{D/2}[e^{\beta\mu}]}-\frac{D^2}{4}\frac{\mathrm{Li}_{D/2}[e^{\beta\mu}]}{\mathrm{Li}_{(D-2)/2}[e^{\beta\mu}]}. \tag{2.10}$$

If we confine ourselves to $D\geqslant 3$, as we do for the rest of the paper, then a critical temperature T_0 can exist at which the chemical potential $\mu=0$. From Eq. (2.7), we find

$$T_0=\left(\frac{2\pi}{m}\right)\left[\frac{N}{V\zeta_R\left(\frac{D}{2}\right)}\right]^{2/D}. \tag{2.11}$$

For $T<T_0$ the chemical potential remains fixed at $\mu=0$. It is easy to see that Eq. (2.10) only holds for $T>T_0$. When $T<T_0$, the specific heat may be evaluated from Eq. (2.8) by setting the term with $(\beta\mu)'$ to zero and also μ to zero. This results in

$$\frac{C_v}{N}=\left(\frac{\beta_0}{\beta}\right)^{D/2}\frac{D(D+2)}{4}\frac{\zeta_R\left(\frac{D+2}{2}\right)}{\zeta_R\left(\frac{D}{2}\right)} \tag{2.12}$$

for $T<T_0$.

The two expressions (2.10) and (2.12) may be used to compute the specific heat for all temperatures, and also to study the behavior at $T=T_0$. It is easy to see from the polylogarithm function (2.4) that if $p\leqslant 1$, $\mathrm{Li}_p[z]\to\infty$ as $z\to 1$. This means that the second term of Eq. (2.10) vanishes as $\mu\to 0$ for $D=3$ and 4, but is finite for $D\geqslant 5$. Thus the specific heat is continuous at $T=T_0$ for $D=3$ and 4, but discontinuous at $T=T_0$ for $D\geqslant 5$. The discontinuity for $D\geqslant 5$ is easily computed in terms of Riemann ζ functions. The continuity of C_v for $D=3$ is well known [18–21]. The behavior for $D>3$ can be found in Refs. [23,24].

III. BOSE–EINSTEIN CONDENSATION IN A CONSTANT EXTERNAL MAGNETIC FIELD

When a magnetic field is applied to a gas of charged bosons in three spatial dimensions the energy spectrum (in the infinite volume limit) contains a discrete harmonic-oscillator-like part as well as a continuous part. The discrete part is just the Landau level quantization [27]. For $D>3$ there may be a number of discrete components because the magnetic field is not described by a vector, but by an antisymmetric tensor with more than one independent component [28]. For simplicity we will restrict our attention to the case of only a single nonzero component in the present paper. We wish to provide a similar treatment to that for the free Bose gas when a nonzero magnetic field is present. In particular we will study the specific heat, and see how the presence of a magnetic field alters the behavior from that found for the free Bose gas in Sec. II. Also we will compute the magnetization and study the Meissner-Ochsenfeld effect in detail. The formalism used is the effective action method as reviewed in Refs. [25,26]. This formalism allows the nonzero condensate (if there is one) to be treated in a very natural manner.

A. Thermodynamic potential and phase transitions: General formalism

The thermodynamic potential is usually defined by

$$\Omega_{T\neq 0}=\frac{1}{\beta}\sum_n\ln[1-e^{\beta(E_n-e\mu)}]. \tag{3.1}$$

However, in the effective action method there is another term present if there is a nonzero condensate described by a background field $\overline{\Psi}$. This is

$$\Omega^{(0)}=\int d^Dx\left\{\frac{1}{2m}|\boldsymbol{D}\overline{\Psi}|^2-e\mu|\overline{\Psi}|^2\right\}. \tag{3.2}$$

Here $\boldsymbol{D}\overline{\Psi}=\boldsymbol{\nabla}\overline{\Psi}-ie\boldsymbol{A}\overline{\Psi}$ is the usual gauge-covariant derivative. The complete thermodynamic potential is

$$\Omega=\Omega^{(0)}+\Omega_{T\neq 0}. \tag{3.3}$$

(Actually, if we are interested in the dynamics of the magnetic field there will be an additional term involving a Maxwell action. We will consider this in Sec. III D below.) Given the thermodynamic potential, all quantities of interest can be calculated.

The presence of a condensate is signalled by a nonzero value for $\overline{\Psi}$. This is associated with symmetry breaking, as discussed in the relativistic case [29,30]. In our case $\overline{\Psi}$ must satisfy

$$\frac{\delta\Omega}{\delta\overline{\Psi}}=0=\frac{1}{2m}\boldsymbol{D}^2\overline{\Psi}+e\mu\overline{\Psi}. \tag{3.4}$$

We can solve this by expanding $\overline{\Psi}(\boldsymbol{x})$ in terms of the stationary state solutions to the Schrödinger equation:

$$\frac{-1}{2m}\boldsymbol{D}^2f_n(\boldsymbol{x})=E_nf_n(\boldsymbol{x}). \tag{3.5}$$

If we write

$$\overline{\Psi}(\boldsymbol{x})=\sum_n C_nf_n(\boldsymbol{x}) \tag{3.6}$$

for some coefficients C_n, and assume that the set of solutions $f_n(\boldsymbol{x})$ forms a complete set, then Eq. (3.4) results in

$$0=(E_n-e\mu)C_n. \tag{3.7}$$

We will define a critical value of μ, say μ_C, by

$$e\mu_C=E_0, \tag{3.8}$$

where E_0 is the lowest energy level. If $\mu<\mu_C$, then the only solution to Eq. (3.7) is for $C_n=0$, which corresponds to $\overline{\Psi}=0$. There is no condensate in this case associated with symmetry breaking and a phase transition. However if μ can reach the value μ_C defined in Eq. (3.8) for some temperature T_C, then C_0 in Eq. (3.7) is undetermined and we can have a nonzero condensate described by

$$\overline{\Psi}(\boldsymbol{x})=C_0f_0(\boldsymbol{x}). \tag{3.9}$$

The temperature T_C at which $\mu=\mu_C$ is called the critical temperature.

For the case of the free gas considered in Sec. II we have $E_0=0$, so that $\mu_C=0$. The critical temperature T_C is then the value of the temperature at which the chemical potential vanishes, as stated earlier. If the spatial dimension $D\geqslant 3$, a critical temperature exists and signals a phase transition with a nonzero value for $\overline{\Psi}$ ($\overline{\Psi}$ is constant for the free Bose gas). Associated with this phase transition is a growth in the number of particles in the ground state.

For some systems it is possible to have a sudden growth in the occupancy of the ground state without a phase transition. In this case μ never reaches the critical value of μ_C, but instead approaches it asymptotically. The speed at which the ground state particle number builds up depends on how fast μ approaches μ_C. Because μ never reaches μ_C we have $\overline{\Psi}=0$, and no symmetry breaking. As we will see below, for a charged Bose gas in a constant magnetic field the spatial dimension D determines whether or not a phase transition occurs.

We can use our expression (3.3) for Ω to find the total charge Q, since $Q=-\partial\Omega/\partial\mu$ with V, B, β, and $\overline{\Psi}$ held fixed. It is convenient to write

$$Q=Q_0+Q_1, \tag{3.10}$$

where

$$Q_0=-\frac{\partial\Omega^{(0)}}{\partial\mu}=e\int d^Dx|\overline{\Psi}|^2=e|C_0|^2, \tag{3.11}$$

if we use Eqs. (3.2) and (3.9), and

$$Q_1=-\frac{\partial\Omega_{T\neq 0}}{\partial\mu}. \tag{3.12}$$

From Eq. (3.1), we find

$$Q_1=e\sum_n\frac{1}{[e^{\beta(E_n-e\mu)}-1]}. \tag{3.13}$$

If we can always solve $Q=Q_1$ for μ for all temperatures, then $\overline{\Psi}=0$. There is no condensate, symmetry breaking, or phase transition in this case. If it is not possible to solve $Q=Q_1$ for μ, then we must have $Q_0\neq 0$ and find a nonzero value for $\overline{\Psi}$.

B. Thermodynamic potential: Constant magnetic field

The formalism outlined in Sec. III A will now be applied to the D-dimensional charged Bose gas in a constant one-component magnetic field. We will assume $D\geqslant 3$ here, and pick the magnetic field in the z direction. It is possible to solve Eq. (3.5) for the energy levels and corresponding eigenfunctions [27]. We have (choosing $\hbar=c=1$)

$$E_{n,k_i}=\left(n+\frac{1}{2}\right)\omega+\frac{1}{2m}\sum_{i=3}^{D}\left(\frac{2\pi k_i}{L_i}\right)^2, \tag{3.14}$$

where $n=0,1,\ldots$ labels the Landau level, and $k_i=0,\pm 1,\ldots$ if we impose periodic boundary conditions on a box as in Sec. II. We have defined

$$\omega=\frac{eB}{m}. \tag{3.15}$$

The energy level (3.14) is degenerate with degeneracy

$$g=\frac{eBL_1L_2}{2\pi}. \tag{3.16}$$

As in Sec. II we will be interested in the large box limit with $L_i\to\infty$. In this limit we can replace the sums over k_i resulting when Eq. (3.14) is used in Eq. (3.1) with integrals. A change of variables gives

$$\Omega_{T\neq 0}=\frac{eBV}{2\pi\beta}\sum_{n=0}^{\infty}\int\frac{d^{D-2}k}{(2\pi)^{D-2}} \times\ln\{1-e^{-\beta[(n+1/2)\omega+(k^2/2m)-e\mu]}\}. \tag{3.17}$$

This may be evaluated by expanding the logarithm in its Taylor series, and then performing the integral over k. We find

$$\Omega_{T\neq 0}=-\omega V\left(\frac{m}{2\pi\beta}\right)^{D/2}\sum_{l=1}^{\infty}\frac{l^{-D/2}e^{-l\beta[(\omega/2)-e\mu]}}{(1-e^{-l\beta\omega})}. \tag{3.18}$$

At this stage it is useful to define a dimensionless temperature. We define

$$x=\beta\omega. \tag{3.19}$$

x is seen to be the ratio between the energy gap between successive energy levels ω and the thermal energy k_BT. The lowest energy level from Eq. (3.14) is $E_{0,0}=\omega/2$. From Eq. (3.8), the critical value for μ is

$$e\mu_C=\frac{\omega}{2}. \tag{3.20}$$

We will define a dimensionless chemical potential ε by

$$e\mu=\omega\left(\frac{1}{2}-\varepsilon\right). \tag{3.21}$$

A phase transition is characterized by a critical temperature T_C at which $\varepsilon=0$. Expression (3.18) may be written in terms of the dimensionless variables x and ε:

$$\Omega_{T\neq 0}=-\omega V\left(\frac{m}{2\pi\beta}\right)^{D/2}\sum_{l=1}^{\infty}\frac{l^{-(D/2)}e^{-l\varepsilon x}}{(1-e^{-lx})}. \tag{3.22}$$

It is convenient to introduce some compact notation for the class of sums we will encounter in order to simplify formulas. Let

$$\Sigma_\kappa[\alpha,\delta]=\sum_{l=1}^{\infty}\frac{l^{\alpha/2}e^{-lx(\varepsilon+\delta)}}{(1-e^{-lx})^\kappa}. \tag{3.23}$$

With this notation we may write Eq. (3.22) as

$$\Omega_{T\neq 0}=-\omega V\left(\frac{m}{2\pi\beta}\right)^{D/2}\Sigma_1[-D,0]. \tag{3.24}$$

Various thermodynamic quantities involve derivatives of the thermodynamic potential. We will initially consider the charge.

From Eq. (3.12), using Eq. (3.24), we have

$$Q_1=\omega V\left(\frac{m}{2\pi\beta}\right)^{D/2}\frac{\partial}{\partial\mu}\Sigma_1[-D,0] =eV\left(\frac{m}{2\pi\beta}\right)^{D/2}x\Sigma_1[2-D,0]. \tag{3.25}$$

Whether or not a phase transition occurs is determined by the convergence or divergence of $\Sigma_1[2-D,0]$ as $\varepsilon\to 0$. If this sum diverges as $\varepsilon\to 0$, then we will always be able to solve $Q=Q_1$ for μ. As discussed in Sec. III A this means that there is no phase transition. From Eq. (3.23) it is easy to see that $\Sigma_1[\alpha,0]$ diverges as $\varepsilon\to 0$ for $\alpha\geqslant -2$. With $\alpha=2-D$, this means that there is no phase transition for $D=3$ and 4 dimensions. For $D\geqslant 5$, $\Sigma_1[2-D,0]$ converges as $\varepsilon\to 0$, and there is a phase transition. In this case (i.e., for $D\geqslant 5$) there is a nonzero condensate characterized by $\overline{\Psi}\neq 0$.

For $D\geqslant 5$ we may use Eq. (3.9) to find $\overline{\Psi}$. If we choose the gauge

$$A_1=-By,\quad A_2=\cdots=A_D=0 \tag{3.26}$$

for the vector potential, then

$$f_0=\alpha e^{-(1/2)eBy^2} \tag{3.27}$$

is the eigenfunction corresponding to the lowest energy $E_0=\omega/2$. α is a constant chosen to normalize f_0. $\Omega^{(0)}$ is found by using Eq. (3.9) with Eq. (3.27) in the general expression (3.2). We will return to this when we discuss the magnetization in Sec. III D.

C. Specific heat

The internal energy is given by $U=(\partial/\partial\beta)(\beta\Omega)$ with V, ω, and $\beta\mu$ held fixed. Using Eqs. (3.1)–(3.3), we find

$$U=\int d^Dx\left\{\frac{1}{2m}|\boldsymbol{D}\overline{\Psi}|^2\right\}+\frac{\partial}{\partial\beta}(\beta\Omega_{T\neq 0}). \tag{3.28}$$

The first term accounts for any nonzero condensate, and the second term is easily seen to be

$$\frac{\partial(\beta\Omega_{T\neq 0})}{\partial\beta}=\sum_n\frac{E_n}{[e^{\beta(E_n-e\mu)}-1]}, \tag{3.29}$$

which is the usual expression for the internal energy [see Eq. (2.2) with a different definition for μ]. With Eqs. (3.5) and (3.9), we find

$$U=E_0|C_0|^2+\frac{\partial}{\partial\beta}(\beta\Omega_{T\neq 0}). \tag{3.30}$$

If $E_0=0$, then the contribution from $\Omega^{(0)}$ to the internal energy vanishes, and the internal energy is given by the standard expression (3.29). This is the situation for the free Bose gas discussed in Sec. II. For the constant magnetic field, $E_0=\omega/2$, so that if $\overline{\Psi}\neq 0$ we must include the condensate contribution to obtain the correct expression for the energy.

If we use Eq. (3.11), then Eq. (3.30) may be written as

$$U=\frac{\omega Q_0}{2e}+\frac{\partial}{\partial\beta}(\beta\Omega_{T\neq0}). \tag{3.31}$$

With Eq. (3.24) for $\Omega_{T\neq0}$, and differentiating with respect to β, keeping $\beta\mu$, ω, and V fixed, results in

$$\frac{\partial(\beta\Omega_{T\neq0})}{\partial\beta}=\omega V\left(\frac{m}{2\pi\beta}\right)^{D/2}\left\{x\Sigma_2[2-D,1]+\frac{x\Sigma_1[2-D,0]}{2}+\frac{(D-2)\Sigma_1[-D,0]}{2}\right\}. \tag{3.32}$$

Noting Eq. (3.25) allows us to write this as

$$\frac{\partial}{\partial\beta}(\beta\Omega_{T\neq0})=\frac{\omega Q_1}{2e}+\omega V\left(\frac{m}{2\pi\beta}\right)^{D/2}\left\{x\Sigma_2[2-D,1]+\left(\frac{(D-2)}{2}\right)\Sigma_1[-D,0]\right\}. \tag{3.33}$$

Substitution of Eq. (3.33) back into Eq. (3.31), and noting from Eq. (3.10) that $Q=Q_0+Q_1$, where Q is the total charge, results in

$$U=\frac{\omega Q}{2e}+\omega V\left(\frac{m}{2\pi\beta}\right)^{D/2}\left\{x\Sigma_2[2-D,1]+\left(\frac{(D-2)}{2}\right)\Sigma_1[-D,0]\right\}. \tag{3.34}$$

The expression for the internal energy we have just obtained [Eq. (3.34)] holds whether there is a nonzero condensate ($\overline{\Psi}\neq0$) or not. In cases where a phase transition does occur, the role of the condensate is crucial for obtaining the correct expression for the internal energy. If we neglected the contribution coming from $\Omega^{(0)}$, then Eq. (3.34) would have Q_1 in place of Q. This would then lead to an erroneous expression for the specific heat, since Q is fixed whereas Q_1 is not.

The specific heat at constant volume was defined in Eq. (2.6). The quantities held fixed are V, Q, $\overline{\Psi}$, and $\omega=eB/m$ when the differentiation is performed. This means that the first term in Eq. (3.34) makes no contribution to C_v. Just like the free Bose gas discussed in Sec. II, we must distinguish between the expressions above and below the the critical temperature if there is a phase transition. For $D\geqslant5$, the critical temperature T_C is defined by

$$Q=eV\left(\frac{m}{2\pi\beta_C}\right)^{D/2}x_C\,\Sigma_1[2-D,0]\big|_{\substack{\varepsilon=0,\\ x=x_C}}. \tag{3.35}$$

Here $x_C=\beta_C\omega$ with $\beta_C=T_C^{-1}$, and the sum $\Sigma_1[2-D,0]$ is evaluated with $\varepsilon=0$ and $x=x_C$. Unlike for the free Bose gas, it is not possible to solve for T_C analytically. We will return to an approximate evaluation of T_C in Sec. IV. We have solved Eq. (3.35) numerically to find T_C. The results for small values of ω are shown in Fig. 1. As expected, when $\omega\to0$ we have $T_C\to T_0$.

For small values of ω, β_C is close to β_0 (an approximate analytical expression will be given in Sec. IV). As the strength of the magnetic field is increased, the deviation of T_C from T_0 becomes more pronounced, but lessens with increasing dimension. In all cases where a phase transition occurs, the critical temperature is lower than the result for the free Bose gas as observed in Ref. [31].

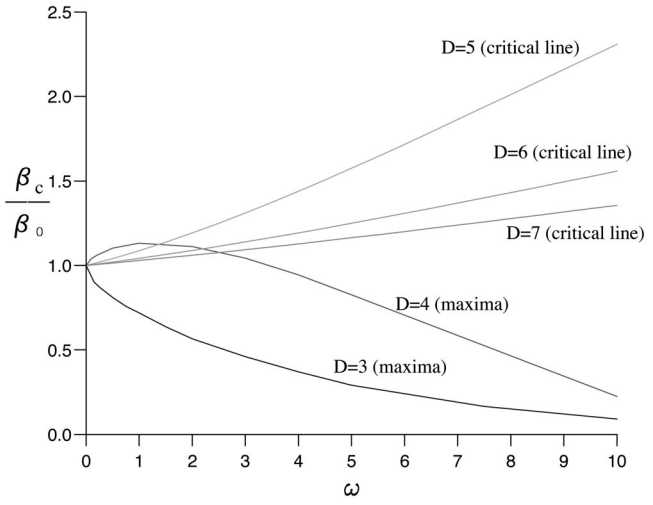

FIG. 1. For $D\geqslant5$, this shows the value of β_C compared to the free Bose gas critical value of β_0. For $D=3$ and 4, because there is no critical temperature when a magnetic field is present, we have plotted the inverse temperature of the specific heat maximum in units of β_0.

Let $C_v^{>}$ denote the specific heat for $T>T_C$, and $C_v^{<}$ be the specific heat for $T<T_C$. Using Eq. (3.34) we have, after some calculation,

$$C_v^{>}=xV\left(\frac{m}{2\pi\beta}\right)^{D/2}\left\{2x^2\Sigma_3[4-D,2]+x^2\Sigma_2[4-D,1]+(D-2)x\Sigma_2[2-D,1]+\frac{D(D-2)}{4}\Sigma_1[-D,0]-\frac{x(D-2)\Sigma_2[4-D,1]\Sigma_1[2-D,0]}{\Sigma_1[4-D,0]}-\frac{x^2(\Sigma_2[4-D,1])^2}{\Sigma_1[4-D,0]}-\left(\frac{D-2}{2}\right)^2\frac{(\Sigma_1[2-D,0])^2}{\Sigma_1[4-D,0]}\right\}. \tag{3.36}$$

For $T<T_C$ we have $\varepsilon=0$ fixed. (Equivalently, $\mu=\omega/2$ is fixed.) The result of this is that $C_v^{<}$ is given by

$$C_v^{<}=xV\left(\frac{m}{2\pi\beta}\right)^{D/2}\left\{2x^2\Sigma_3[4-D,2]+x^2\Sigma_2[4-D,1]+(D-2)x\Sigma_2[2-D,1]+\frac{D(D-2)}{4}x\Sigma_1[-D,0]\right\}\Bigg|_{\varepsilon=0}. \tag{3.37}$$

By comparing $C_v^{>}$ in Eq. (3.36) with $C_v^{<}$ in Eq. (3.37), it can be seen that whether or not the specific heat is continuous at the critical temperature is determined by the behavior of $\Sigma_1[4-D,0]$ as $\varepsilon\to0$. The two expressions will only agree if $\Sigma_1[4-D,0]$ diverges in this limit. From definition (3.23) this only happens for $D=5$ and 6 (recall that we are assuming $D\geqslant5$ here so that T_C exists). We conclude that the spe-

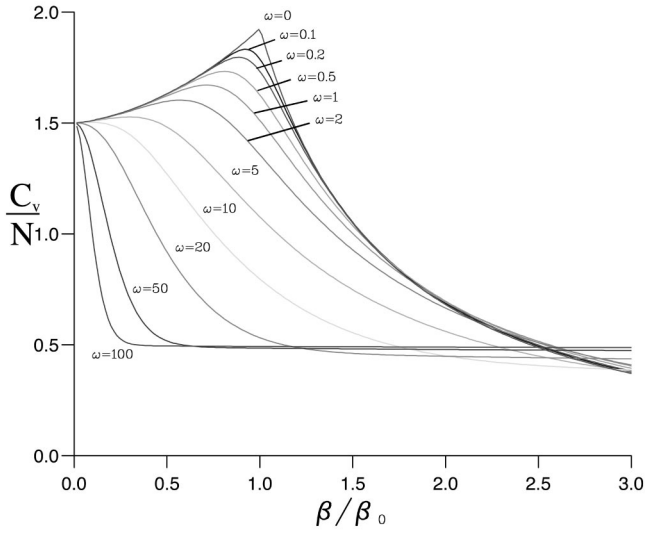

FIG. 2. This shows the specific heat at constant volume in units of the total number of particles as a function of the inverse temperature in units of β_0 where $\beta_0 = T_0^{-1}$, with T_0 given in Eq. (2.11) for $D=3$. For comparison, the free gas result is also shown ($\omega = 0$).

cific heat for the charged Bose gas in a constant magnetic field is continuous at the critical temperature for $D=5$ and 6, and discontinuous for $D \geqslant 7$.

In the cases $D=3$ and 4, where there is no phase transition, ε never vanishes. The specific heat is given by Eq. (3.36) in these two cases for all temperatures. When $D=3$ and 4 the specific heat is a perfectly smooth function of temperature.

Graphs showing the specific heat for $D=3$ and 5 are shown in Figs. 2 and 3. ($D=5$ is chosen to illustrate the higher dimensional results, as one example where the free-field gas has a discontinuous specific heat.)

For $D=3$, the presence of the magnetic field is seen in Fig. 2 to round off the familiar sharp behavior at the transition temperature for the free gas. (The result in $D=4$ is qualitatively similar.) As the magnetic field is reduced the curves for the specific heat tend toward the free-field result. The maximum in the specific heat becomes sharper, and the temperature at which the maximum occurs tends toward the value T_0 as the magnetic field is reduced. For $B \neq 0$, although there is no phase transition characterized by a critical temperature and a nonzero condensate, as B is reduced the specific heat starts to look more and more like the free gas result. We will look at this analytically in Sec. IV. As $\beta \rightarrow 0$ (or $T \rightarrow \infty$) the specific heat for $D=3$ approaches the classical Maxwell-Boltzmann result of 1.5.

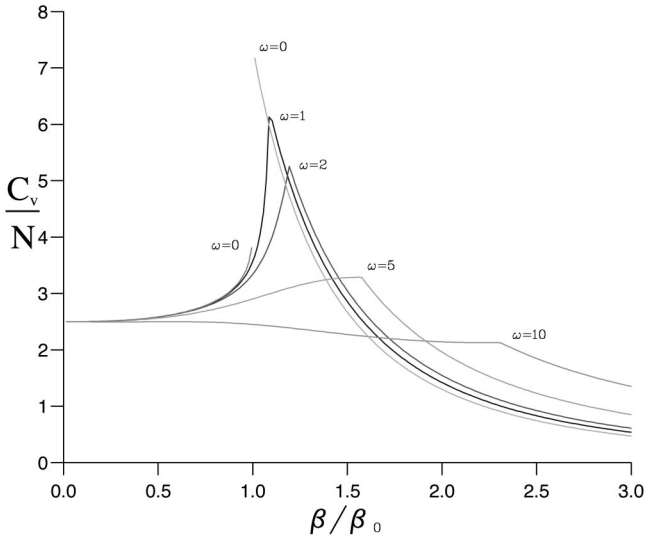

FIG. 3. The specific heat at constant volume in units of the total number of particles for the magnetized gas with $D=5$. The curves are given as a function of $\beta = T^{-1}$ in units of β_0 rather than β_C. The discontinuous free gas result is labeled $\omega = 0$.

The behavior of the specific heat as B is increased is also of interest. Increasing the value of B tends to reduce the specific heat maximum and to broaden the curves. In fact, for large values of B, if we examine Fig. 2 it can be seen that the curves approach the value of $\frac{1}{2}$ as B is reduced before rising sharply to the classical result of $\frac{3}{2}$. This demonstrates that the specific heat for the gas with $D=3$ resembles the specific heat for the free gas with $D=1$ in a strong magnetic field. The value of $\frac{1}{2}$ is the classical value for the one-dimensional gas. This is totally consistent with the approach used in Refs. [32,33], in which the leading behavior of thermodynamic quantities was studied in a general setting by using the lowest energy solutions. As B is increased, the gap between the ground state and the excited states becomes larger; thus the leading contribution would be expected to come from the ground state.

The results for the specific heat of the five-dimensional gas are shown in Fig. 3. In this case the specific heat for the free Bose gas is discontinuous. Nevertheless, as the magnetic field is reduced the specific heat curves approach the free gas result. The peaks of the specific heat start to become sharper and the slope of the curve steeper. The classical Maxwell-Boltzmann result of $\frac{5}{2}$ is reached as $\beta \rightarrow 0$. Just as for the case $D=3$, the gas exhibits a reduction in the effective dimension for large values of the magnetic field. This time we would expect to find the specific heat curves looking more and more like the familiar form for the specific heat of the free gas in three spatial dimensions, and this is what is found if the numerical results of Fig. 3 are studied closely.

D. Magnetization

Even though for $D=3$ there is no phase transition which can be associated with Bose-Einstein condensation, by studying the magnetization of the charged gas Schafroth [2] showed that the gas exhibited the Meissner-Ochsenfeld effect. The generalization to other spatial dimensions was performed later [3–10,33]. We will show how the formalism described in Sec. III A can be used to obtain the magnetization. In particular, the role of the condensate $\overline{\Psi}$ for $D \geqslant 5$ will be examined carefully.

The simplest way to see the effects of magnetization is by studying how the field equations for electromagnetism are affected. To do this we must include a term in the thermodynamic potential (3.3) for the electromagnetic field. We will use Heaviside-Lorentz rationalized units as usual in quantum field theory. (A discussion of the various units and how this alters the expression for the magnetization was given in Ref. [33]. Of course the physics of the situation

should be independent of this arbitrary choice.) We add

$$\Omega_{\rm em}=\int d^Dx\left(\frac{1}{4}F_{ij}F^{ij}-J^i_{\rm ext}A_i\right) \tag{3.38}$$

to Eq. (3.3), where $F_{ij}=\partial_iA_j-\partial_jA_i$ is the field strength tensor describing the magnetic field, and $J^i_{\rm ext}$ is the externally applied current which is responsible for setting up the magnetic field. The complete thermodynamic potential is

$$\Omega=\Omega_{\rm em}+\Omega^{(0)}+\Omega_{T\neq0} \tag{3.39}$$

where $\Omega^{(0)}$ and $\Omega_{T\neq0}$ are given in Eqs. (3.1) and (3.2).

Variation of Ω with respect to the magnetic field F_{ij} results in

$$\partial_jH^{ij}=J^i_{\rm ext}, \tag{3.40}$$

where

$$H^{ij}=F^{ij}+2\frac{\delta}{\delta F_{ij}}(\Omega^{(0)}+\Omega_{T\neq0}). \tag{3.41}$$

H^{ij} is the D-dimensional analog of the usual vector H in three-dimensional electromagnetism. As explained earlier it is necessary to treat the magnetic field as a tensor if $D\neq3$. For more details of this analysis, see Ref. [26].

We have treated the magnetic field generally in Eqs. (3.38)–(3.41). Specializing now to a single component field of strength $B(F_{12}=-F_{21}=B)$, Eq. (3.41) can be written as

$$H=B-M, \tag{3.42}$$

where

$$M=-\frac{\delta}{\delta B}(\Omega^{(0)}+\Omega_{T\neq0}), \tag{3.43}$$

and $H^{12}=-H^{21}=H$. Equation (3.42) is the conventional B-H relation found in three spatial dimensions, but with the notation defined here it can be seen to hold for all D. M in Eq. (3.43) is the magnetization. This approach is seen to avoid any ambiguity between what Schafroth called the acting and microscopic fields.

We can now split M in Eq. (3.43) into two pieces in an obvious way. The derivative in Eq. (3.43) is a functional derivative, and because B is a constant for our problem we can define

$$M^{(0)}=-\frac{1}{V}\frac{\partial\Omega^{(0)}}{\partial B}, \tag{3.44}$$

$$M_{T\neq0}=-\frac{1}{V}\frac{\partial\Omega_{T\neq0}}{\partial B}. \tag{3.45}$$

$M_{T\neq0}$ is easily computed, using Eq. (3.24), to be

$$M_{T\neq0}=\frac{e}{m}\left(\frac{m}{2\pi\beta}\right)^{D/2}\left\{\Sigma_1[-D,0]-\frac{x}{2}\Sigma_1[2-D,0]-x\Sigma_2[2-D,1]\right\}. \tag{3.46}$$

Recalling Eq. (3.25), we have

$$M_{T\neq0}=-\frac{1}{2m}\frac{Q_1}{V}+\frac{e}{m}\left(\frac{m}{2\pi\beta}\right)^{D/2}\times\{\Sigma_1[-D,0]-x\Sigma_2[2-D,1]\}. \tag{3.47}$$

In cases where no phase transition occurs, we have $Q_1=Q$, so that the first term of Eq. (3.47) is constant. Also if $\overline{\Psi}=0$ then $M^{(0)}=0$, so that the total magnetization is given by Eq. (3.47) with $Q_1=Q$.

When a phase transition does occur we need $M^{(0)}$. With our gauge choice [Eq. (3.26)], and using Eq. (3.2) for $\Omega^{(0)}$, we find

$$M^{(0)}=-\frac{e^2B}{mV}\int d^Dxy^2|\overline{\Psi}|^2. \tag{3.48}$$

$\overline{\Psi}$ was given by Eq. (3.9) with Eq. (3.27). It is easy to show that

$$\int d^Dxy^2|f_0|^2=\frac{1}{2eB}, \tag{3.49}$$

which results in

$$M^{(0)}=-\frac{e}{2mV}|C_0|^2=-\frac{Q_0}{2mV} \tag{3.50}$$

after using Eq. (3.11). We may now combine Eqs. (3.47) and (3.50) to read

$$M=-\frac{Q}{2mV}+\frac{e}{m}\left(\frac{m}{2\pi\beta}\right)^{D/2}\{\Sigma_1[-D,0]-x\Sigma_2[2-D,1]\}, \tag{3.51}$$

since $Q=Q_0+Q_1$ is the total charge.

The result in Eq. (3.51) is the exact expression for the magnetization, which holds even if there is a phase transition. Had the condensate $\overline{\Psi}$ been ignored, we would have obtained Eq. (3.47) rather than Eq. (3.51). In the true expression (3.51) the first term is constant, whereas in Eq. (3.47) the first term is not constant if there is a phase transition. Thus neglect of the condensate for $D\geqslant5$ would lead to an erroneous result for the magnetization.

The dimensionless magnetization $\mathcal{M}=mVM/Q$ is shown for $D=3$ in Fig. 4. The zero-field spontaneous magnetization plotted on the graphs is of the form given originally by Schafroth [2],

$$M=-\frac{Q}{2mV}\left[1-\left(\frac{T}{T_0}\right)^{D/2}\right], \tag{3.52}$$

(with $D=3$) and it can be seen that as the field $B\to0$, this limit is recovered. As the number of dimensions increases, the diamagnetism due to macroscopic occupation of the lowest Landau level in the low temperature (high β/β_0) region becomes more pronounced. The general behavior of the magnetized gas is similar in cases when a phase transition is absent ($D<5$) or present ($D\geqslant5$).

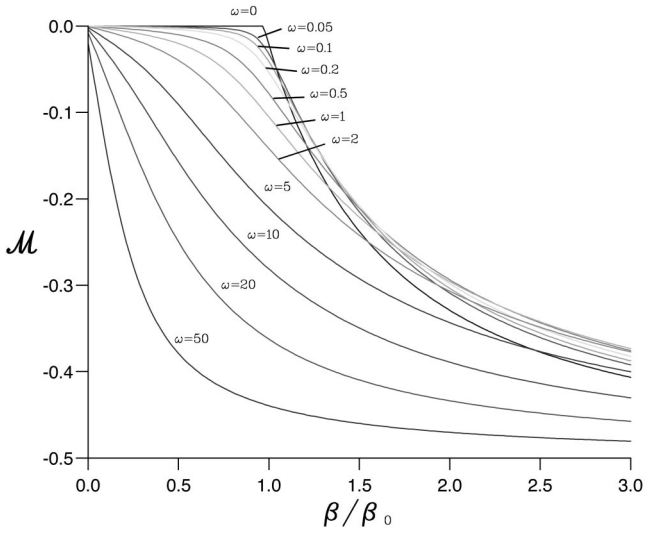

FIG. 4. The dimensionless magnetization $\mathcal{M}=mVM/Q$ for the charged Bose gas in three spatial dimensions.

IV. ANALYTICAL EXPANSIONS

So far we have presented mainly the results obtained from numerical evaluations of sum (3.23), since it is not possible to obtain exact results. However due to the form of the exponential involving e^{-lx}, in order to obtain reliable numerical results directly from Eq. (3.23) it is necessary to include an increasing number of terms as x decreases. In this section we will discuss a reliable method for obtaining approximate analytical expressions for various situations when ε and x are small by finding asymptotic expansions for $\Sigma_\kappa[\alpha,\delta]$.

The basic method we will use here involves the Mellin-Barnes contour integral representation for the exponential function:

$$e^{-v}=\int_{c-i\infty}^{c+i\infty}\frac{d\theta}{2\pi i}\Gamma(\theta)v^{-\theta}. \tag{4.1}$$

Here c is a constant with $\mathrm{Re}c>0$, so that the contour lies to the right of the poles of the gamma function. This is essentially equivalent to the method used by Robinson [34] to obtain asymptotic expansions for the Bose-Einstein functions. It was used for the three-dimensional magnetized gas by Daicic and Frankel [9], and was used to discuss Bose-Einstein condensation in a harmonic oscillator potential in Ref. [17]. There are various ways in which Eq. (4.1) can be used to obtain expansions over a range of x and ε, as discussed in Refs. [17,26]. We will content ourselves with the simplest presentation here.

The basic method is to use Eq. (4.1) to convert Eq. (3.23) into a contour integral. The contour may be closed in the left hand side of the complex plane, and the result evaluated by the residue theorem. The even and odd spatial dimensions differ somewhat in the pole structure of the integrand. The net result is an asymptotic series for $\Sigma_\kappa[\alpha,\delta]$ which can be used to approximate the specific heat, magnetization, and other thermodynamic quantities. The details of these calculations are lengthy and only the relevant results will be presented here. The reader interested in details can consult Refs. [35,36].

A. Critical temperature

For $D\geqslant 5$ the magnetized Bose gas is characterized by a well defined critical temperature T_C which satisfies Eq. (3.35). It is not possible to evaluate T_C in a closed form. However, we can use our asymptotic expansion of $\Sigma_1[2-D,0]$ to obtain an approximate result for weak magnetic fields. We have

$$Q\simeq eV\left(\frac{m}{2\pi\beta_C}\right)^{D/2}\left\{\zeta_R\left(\frac{D}{2}\right)+\frac{x_C}{2}\zeta_R\left(\frac{D-2}{2}\right)+\cdots\right\} \tag{4.2}$$

if only the two leading terms are included. This assumes $x_C\ll 1$. With the free Bose gas critical temperature defined by

$$Q=eV\left(\frac{m}{2\pi\beta_0}\right)^{D/2}\zeta_R\left(\frac{D}{2}\right), \tag{4.3}$$

we find

$$0\simeq\left[1-\left(\frac{\beta_C}{\beta_0}\right)^{D/2}\right]\zeta_R\left(\frac{D}{2}\right)+\frac{x_C}{2}\zeta_R\left(\frac{D-2}{2}\right). \tag{4.4}$$

Since x_C is assumed small we see that $\beta_C\simeq\beta_0$. It is easy to show from Eq. (4.4) that

$$T_C\simeq T_0-\frac{1}{D}\frac{\zeta_R\left(\frac{D-2}{2}\right)}{\zeta_R\left(\frac{D}{2}\right)}\frac{eB}{m} \tag{4.5}$$

to leading order in eB/m. This shows that $T_C\to T_0$ as $B\to 0$. Furthermore, for a fixed charge density, the critical temperature is lower when a nonzero magnetic field is present. This is consistent with our earlier numerical results.

A cruder estimate of T_C was given in Ref. [31], which had the same linear behavior as in Eq. (4.5) but with a different numerical factor in front of B. Our result [Eq. (4.5)] is a special case of the multicomponent magnetic field presented in Ref. [26].

It is possible to improve on the linear approximation of Eq. (4.5) by working consistently to higher order in the expansions. It is necessary to deal with $D=5$ and 6 separately from $D>6$ because of the order of the terms retained. For $D=5$, we find

$$\frac{T_C}{T_0}\simeq 1-\frac{1}{5}\frac{\zeta_R\left(\frac{3}{2}\right)}{\zeta_R\left(\frac{5}{2}\right)}x_0-\frac{1}{5\pi^{1/2}}\frac{\zeta_R\left(\frac{3}{2}\right)}{\zeta_R\left(\frac{5}{2}\right)}x_0^{3/2}+\mathcal{O}(x_0^2). \tag{4.6}$$

It is worth remarking that the result given by May [4] is only correct if it is taken to linear order in B.

We can also obtain an approximate expression for the charge in the condensate when $T\leqslant T_C$ for $D\geqslant 5$. When $T\leqslant T_C$ we have $\varepsilon=0$, so Eq. (3.25) gives us

$$Q_1=eV\left(\frac{m}{2\pi\beta}\right)^{D/2}x\Sigma_1[2-D,0]\big|_{\varepsilon=0}. \tag{4.7}$$

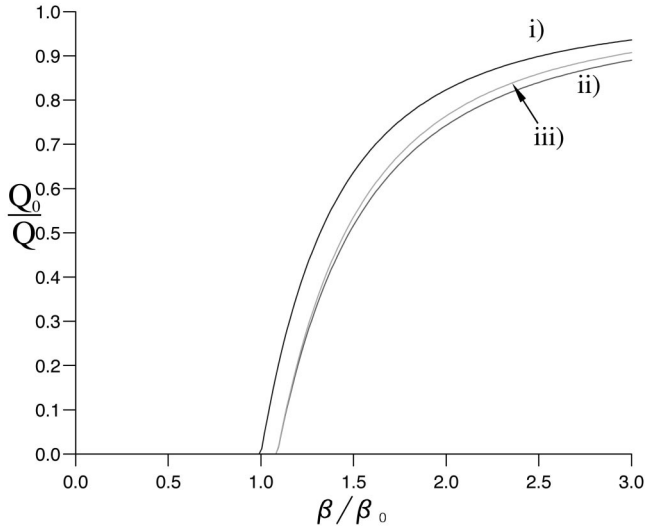

FIG. 5. The ratio of the charge contained in the ground state over the total charge for (i) the free gas [Eq. (4.11)], (ii) the exact result (4.9), and (iii) the approximation (4.10) (for $D=5$ and $\omega=1$).

By using Eq. (3.35), which defines the critical temperature T_C, we find (noting $x=\beta\omega$, $x_C=\beta_C\omega$)

$$Q_1=Q\left(\frac{x_C}{x}\right)^{(D-2)/2}\frac{\Sigma_1[2-D,0]|_{\varepsilon=0}}{\Sigma_1[2-D,0]|_{\substack{\varepsilon=0,\\ x=x_C}}}. \tag{4.8}$$

From Eq. (3.10), we find that the charge in the condensate is

$$Q_0=Q\left\{1-\left(\frac{x_C}{x}\right)^{(D-2)/2}\frac{\Sigma_1[2-D,0]|_{\varepsilon=0}}{\Sigma_1[2-D,0]|_{\substack{\varepsilon=0\\ x=x_C}}}\right\}. \tag{4.9}$$

This result is exact. If we now use the approximate analytical expressions for Σ_1, we obtain

$$Q_0=Q\left\{1-\left(\frac{x_C}{x}\right)^{D/2}-\frac{1}{2}\left(\frac{x_C}{x}\right)^{D/2}\frac{\zeta_R\left(\frac{D-2}{2}\right)}{\zeta_R\left(\frac{D}{2}\right)}(x-x_C)\right\}. \tag{4.10}$$

It is worth remarking that the accuracy of any of our approximate expressions can be increased by simply including more terms.

From Eq. (4.10), we see that as $\omega\to 0$ because $x_C/x=T/T_C$, and we know that $T_C\to T_0$, we recover the free Bose gas result

$$Q_0=Q\left[1-\left(\frac{T}{T_0}\right)^{D/2}\right]. \tag{4.11}$$

The term in Eq. (4.10) which involves $x-x_C$ represents the lowest order correction to the free field result in a weak magnetic field.

In Fig. 5 we plot the free gas result [Eq. (4.11)], the exact numerical result, and our approximation (4.10) for the case $D=5$. It can be seen that Eq. (4.10) is very close to the true result.

B. Magnetization for $D\geqslant 5$

The magnetization may be evaluated using Eq. (3.51). It is necessary to distinguish the cases $D=3$ and 4 for which no phase transition occurs from $D\geqslant 5$. We will deal with $D\geqslant 5$ in this section, leaving $D=3$ and 4 until later. With $D\geqslant 5$ we must deal with $T\geqslant T_C$ and $T\leqslant T_C$ since the chemical potential is different in the two cases. For simplicity we will only give the expressions for $T\leqslant T_C$. When $T\geqslant T_C$ we have $\varepsilon\neq 0$ and the results are more complicated. In addition, the expansions break down once ε grows too large.

For $T\leqslant T_C$ we use Eq. (3.51) with $\varepsilon=0$:

$$M=-\frac{Q}{2mV}+\frac{e}{m}\left(\frac{m}{2\pi\beta}\right)^{D/2}\times\{\Sigma_1[-D,0]-x\Sigma_2[2-D,1]\}|_{\varepsilon=0}. \tag{4.12}$$

This can be rewritten if we eliminate $(m/2\pi)^{D/2}$ using Eq. (3.35). We find

$$M=-\frac{Q}{2mV}\left\{1-2\left(\frac{T}{T_C}\right)^{D/2}\times\frac{\Sigma_1[-D,0]-x\Sigma_2[2-D,1]|_{\varepsilon=0}}{x_C\Sigma_1[2-D,0]|_{\substack{\varepsilon=0,\\ x=x_C}}}\right\} \tag{4.13}$$

It is now a straightforward but tedious matter to expand the sums.

For $D=5$ the asymptotic expansion of the magnetization is given by

$$M=-\frac{Q}{2mV}\left\{1-\left(\frac{T}{T_C}\right)^{5/2}-\left(\frac{T}{T_C}\right)^{5/2}\times\left(\frac{x}{3}-\frac{x_C}{2}\right)\frac{\zeta_R\left(\frac{3}{2}\right)}{\zeta_R\left(\frac{5}{2}\right)}+\cdots\right\}. \tag{4.14}$$

By taking $B\to 0$ we are left with the spontaneous magnetization

$$M(B\to 0)\simeq-\frac{Q}{2mV}\left[1-\left(\frac{T}{T_0}\right)^{5/2}\right] \tag{4.15}$$

since $T_C\to T_0$ in this limit. This is the five-dimensional version of Schafroth's result, and agrees with May [4]. When $B\neq 0$ there are corrections to the Schafroth form as shown by the third term in Eq. (4.14). There are two sources for these corrections. The first is that for $B\neq 0$ and $T_C\neq T_0$. The second is that the asymptotic form of M has higher order terms present. The fact that M does not vanish as $B\to 0$ shows that the Meissner-Ochsenfeld effect exists.

We can also compare our result with that given by May [4]. To do this we must replace T_C in Eq. (4.14) with T_0. We will only work to first order in the magnetic field and use Eq. (4.14). It is easily shown that

$$M \simeq -\frac{Q}{2mV}\left\{1-\left(\frac{T}{T_0}\right)^{5/2}-\frac{1}{3}\left(\frac{T}{T_0}\right)^{5/2}\frac{\zeta_R\left(\frac{3}{2}\right)}{\zeta_R\left(\frac{5}{2}\right)}x+\cdots\right\}. \tag{4.16}$$

This agrees with May's result, where a different method was used. There are two comments to make here. The first is that had we taken the expansion beyond linear order in x the results obtained would not agree with that of May's method for reasons already mentioned. Second, care must be exercised in using Eq. (4.16) because we have assumed $T \leqslant T_C$ in its derivation. This means that Eq. (4.16) does not hold at $T=T_0$, as can be seen from Eq. (4.5). If we set $T=T_0$ in Eq. (4.16), we would conclude that M was positive, leading to paramagnetic rather than diamagnetic behavior. This is clearly wrong. We will return to results for $T=T_0$ later. There is, of course, nothing wrong with taking $T=T_C$ in Eq. (4.14).

C. Specific heat

In a similar manner to Sec. IV B, analytical expansions may be obtained for the specific heat capacities in the dimensions of interest to us in this paper. It is more convenient to express these results in the form they were given previously, i.e., as expansions of the ratio C_v/N. These expansions are constructed from the expression for the heat capacity [either Eq. (3.36) if above the transition temperature, or Eq. (3.37) if below it] as well as that for the number density [obtained from Eq. (3.25) as $N=Q/e$] .

For $D=3$ from Fig. 2, it can be seen that the charged Bose gas in three spatial dimensions does not exhibit a phase transition with finite field, but approaches the zero field result in the $B\to 0$ limit. The maximum of the heat capacity is always lower than the zero field limit, a fact which can clearly be seen from the analytic expansion

$$\frac{C_v}{N} \simeq \frac{15\zeta_R\left(\frac{5}{2}\right)}{4\zeta_R\left(\frac{3}{2}\right)} - \frac{3x^{1/2}}{4\sqrt{\pi}\zeta_R^2\left(\frac{3}{2}\right)\zeta_H\left(\frac{3}{2},\varepsilon\right)}\left[6\zeta_R^3\left(\frac{3}{2}\right) + 5\pi\zeta_R\left(\frac{5}{2}\right)\zeta_H\left(\frac{1}{2},\varepsilon\right)\zeta_H\left(\frac{3}{2},\varepsilon\right)\right] + O(x).$$

The five-dimensional gas is the first to show the existence of a phase transition, and as previously discussed in Sec. III C, it is necessary to use different expressions for the heat capacity above and below the critical temperature. For $T > T_C$, this quantity is given by

$$\frac{C_v}{N} \simeq \frac{35\zeta_R\left(\frac{7}{2}\right)\zeta_R\left(\frac{3}{2}\right)-25\zeta_R^2\left(\frac{5}{2}\right)}{4\zeta_R\left(\frac{3}{2}\right)\zeta_R\left(\frac{5}{2}\right)} + \frac{25\pi^{1/2}x^{1/2}\,\zeta_H\left(\frac{1}{2},\varepsilon\right)\zeta_R\left(\frac{5}{2}\right)}{4\zeta_R^2\left(\frac{3}{2}\right)} + O(x^{3/2}),$$

while below T_C Eq. (3.37) must be used, and hence

$$\frac{C_v}{N} \simeq \frac{35\zeta_R\left(\frac{7}{2}\right)}{4\zeta_R\left(\frac{5}{2}\right)} + \frac{x\left(15\zeta_R^2\left(\frac{5}{2}\right)-35\zeta_R\left(\frac{3}{2}\right)\zeta_R\left(\frac{7}{2}\right)\right)}{8\zeta_R^2\left(\frac{5}{2}\right)} + O(x^{3/2}). \tag{4.17}$$

D. Expansions for T near T_0

As noted in Schafroth's original paper [2] the approximation (for the three-dimensional gas)

$$M \simeq -\frac{Q}{2mV}\left[1-\left(\frac{T}{T_0}\right)^{3/2}\right] \tag{4.18}$$

will break down when T becomes too close to T_0. This can be substantiated by a direct numerical evaluation of M and comparison with Eq. (4.18), as we showed in Ref. [37]. The Schafroth criterion for validity of Eq. (4.18) can be derived in a simple manner, as described in Ref. [9]. An obvious question to ask is that if Eq. (4.18) does not hold as T becomes close to T_0, is there another simple approximation which can be used? This was first studied by Daicic and Frankel [9], who showed by expanding about $\mu=0$ that the magnetization $M \simeq -cB^{1/2}$, with $c>0$ a constant. This approximation was valid for values of T closer to T_0 than Schafroth's approximation, but still broke down as $T\to T_0$. In Ref. [37] we showed how it was possible to evaluate the magnetization in a temperature range which included $T=T_0$. In this section we will present the details of this result, and generalize the analysis to spatial dimensions $D>3$.

For $D=3$ the critical temperature T_0 for the free Bose gas is defined by

$$Q = eV\left(\frac{m}{2\pi\beta_0}\right)^{3/2}\zeta_R\left(\frac{3}{2}\right), \tag{4.19}$$

where $\beta_0 = T_0^{-1}$. For $B\neq 0$ we have

$$Q = eV\left(\frac{m}{2\pi\beta}\right)^{3/2} x\Sigma_1[-1,0]. \tag{4.20}$$

Equating these two expressions gives

$$x\Sigma_1[-1,0] = \left(\frac{x}{x_0}\right)^{3/2}\zeta_R\left(\frac{3}{2}\right), \tag{4.21}$$

where $x_0=\beta_0\omega$. The aim now is to solve this for ε when the magnetic field is weak and T is close to T_0 (meaning x is

close to x_0). Because we assume a weak magnetic field, we may use the first few terms in the asymptotic expansion of $\Sigma_1[-1,0]$:

$$\left(\frac{x}{x_0}\right)^{3/2}\zeta_R\left(\frac{3}{2}\right)\simeq\zeta_R\left(\frac{3}{2}\right)+(\pi x)^{1/2}\zeta_H\left(\frac{1}{2},\varepsilon\right)+\zeta_R\left(\frac{1}{2}\right)\left(\frac{1}{2}-\varepsilon\right)x+\cdots. \quad (4.22)$$

Higher order terms could easily be included to improve the accuracy of the result.

Suppose that we concentrate first on $T=T_0$. Call the value of ε at $T=T_0$, ε_0. Then Eq. (4.22) gives us

$$0=\zeta_H\left(\frac{1}{2},\varepsilon_0\right)+\pi^{-1/2}\zeta_R\left(\frac{1}{2}\right)\left(\frac{1}{2}-\varepsilon_0\right)x_0^{1/2}+\cdots. \quad (4.23)$$

(The next term is of order $x^{3/2}$.) If we let $B\to 0$, then $x_0=\beta_0\omega\to 0$. Thus as $B\to 0$ we must have $\varepsilon_0\to a$, where a is defined by

$$0=\zeta_H\left(\frac{1}{2},a\right). \quad (4.24)$$

The value of a can be found numerically, with the result $a=0.302721829\ldots$. We have verified this by solving Eq. (4.21) numerically for decreasing values of B, and found that $\varepsilon\to a$ as B is reduced. Because $\mu=\omega(\frac{1}{2}-\varepsilon)$ this result is still consistent with the expectation that $\mu\to 0$ as $B\to 0$. We have essentially determined how fast μ vanishes as $B\to 0$.

For small, but nonzero, values of B, we can try to solve Eq. (4.23). In order to obtain a consistent expansion from Eq. (4.23), it is fairly clear that we must have

$$\varepsilon_0\simeq a+a_1x_0^{1/2}+a_2x_0+\cdots \quad (4.25)$$

for some coefficients $a_1, a_2, \ldots$ which can be found by substituting Eq. (4.25) into Eq. (4.23) and working to a consistent order in x_0. It is easily shown that

$$\zeta_H\left(\frac{1}{2},\varepsilon_0\right)\simeq-\frac{1}{2}a_1x_0^{1/2}\zeta_H\left(\frac{3}{2},a\right)+O(x_0). \quad (4.26)$$

Use of Eqs. (4.25) and (4.26) in Eq. (4.23) fixes

$$a_1=\frac{2\zeta_R\left(\frac{1}{2}\right)}{\pi^{1/2}\zeta_H\left(\frac{3}{2},a\right)}\left(\frac{1}{2}-a\right). \quad (4.27)$$

It should be clear how we can obtain an approximation for ε_0 to any order in x_0 by extending the procedure we have just described to higher order.

So far we have just concentrated on the evaluation of ε at the single temperature T_0. Suppose that we now extend this to temperatures which are close to T_0. By a simple extension of the analysis just presented, it is possible to show

$$\varepsilon\simeq a+a_1x_0^{1/2}+\frac{6\zeta_R\left(\frac{3}{2}\right)}{\pi^{1/2}\zeta_H\left(\frac{3}{2},a\right)}x_0^{-1/2}\left[1-\left(\frac{x}{x_0}\right)^{1/2}\right]+\cdots. \quad (4.28)$$

Consistency of this expansion requires

$$1-\left(\frac{x}{x_0}\right)^{1/2}\ll x_0^{1/2}. \quad (4.29)$$

In particular, the approximation is good at $x=x_0$.

The approximation we have found for ε may now be used in the expressions for the specific heat and the magnetization. For the specific heat at $T=T_0$, we find

$$\frac{C_v}{N}\simeq\frac{15\zeta_R\left(\frac{5}{2}\right)}{4\zeta_R\left(\frac{3}{2}\right)}-x_0^{1/2}\frac{9\zeta_R\left(\frac{3}{2}\right)}{2\pi^{1/2}\zeta_H\left(\frac{3}{2},a\right)}+O(x_0). \quad (4.30)$$

This shows that as $B\to 0$ the specific heat approaches the free-field result at $T=T_0$, confirming analytically the trend we found numerically. It also provides an analytic proof that for small x_0, the presence of the magnetic field lowers the value of the specific heat from the free-field value.

For the magnetization we find (at $T=T_0$)

$$M\simeq-\frac{Q}{2mV}\left\{\frac{6\pi^{1/2}\zeta_H\left(-\frac{1}{2},a\right)}{\zeta_R\left(\frac{3}{2}\right)}x_0^{1/2}-2\frac{\zeta_H\left(\frac{1}{2}\right)}{\zeta_R\left(\frac{3}{2}\right)}\left(\frac{1}{6}-a+a^2\right)x_0+\cdots\right\}. \quad (4.31)$$

This provides confirmation of the $M\simeq -cB^{1/2}$ magnetization law of Daicic and Frankel [9], but at a lower temperature (and therefore the coefficient of $B^{1/2}$ differs from that of [9]). In addition, we have computed the next order correction to the leading $B^{1/2}$ behavior. It is straightforward to extend this analysis to cases other than $D=3$ [35,36].

V. DISCUSSION AND CONCLUSIONS

This paper has studied the thermodynamic properties of the ideal charged Bose gas in some detail. Spatial dimensions with $D\geqslant 3$ have been examined, since even for the free gas it is known that the properties of the gas are sensitive to the spatial dimension. The specific heat was calculated numerically as well as analytically, for small values of the magnetic field. We also performed calculations of the magnetization and showed how the effective action method could be used to account for the condensate when $D\geqslant 5$.

One motivation for our study was to understand in more detail the behavior of the magnetized gas for $D=3$. When a magnetic field is present, no matter how small, there is no phase transition; however, when there is no magnetic field

the system does exhibit a phase transition with a nonzero condensate. On physical grounds we would expect that there should be some sense in which the system in a small magnetic field should behave in almost the same way as the free gas. By examining the specific heat it is possible to see that as the magnetic field is reduced, the curves start to resemble the specific heat for the free gas. So long as the magnetic field remains nonzero the specific heat is always smooth, with the specific heat maximum approaching the free Bose gas transition temperature as the magnetic field is reduced. The results of Sec. IV may be used to study this analytically. Although we did not show it, it is straightforward to show that the derivative of the specific heat becomes discontinuous as $B \to 0$, exactly as in the case for the free Bose gas. Similar remarks apply to gases in other dimensions.

We also studied the behavior in large magnetic fields. Here we found that the specific heat for a gas in D spatial dimensions looked like the specific heat for the free gas in $D-2$ spatial dimensions over a range of temperatures. Once the temperature becomes too large, this effective reduction in dimension disappears.

By using the Mellin-Barnes integral transform we were able to obtain a number of analytical approximations. Although, for $D \geqslant 5$, it is not possible to solve for the critical temperature exactly, it is possible to obtain good estimates when the magnetic field is weak. These approximations can then be used to study the Meissner-Ochsenfeld effect. We were also able to obtain reliable approximations valid at the free gas condensation temperature. As a by-product, our calculations showed exactly how the chemical potential vanishes as $B \to 0$.

The most obvious way that the calculations given in this paper should be extended is by the inclusion of Coulomb interactions among the charged particles. Clearly this is essential before it will be possible to study Bose-Einstein condensation of charged particles in a reliable way. Given the recent experimental advances in the cooling and trapping of atomic gases, it may become increasingly important to study this problem for trapped ions. A less pressing extension of our work is to relativistic charged particles. Daicic and coworkers [7,9] performed a study of this already in various cases; however, it would be easily possible to extend the analysis of our paper to obtain the specific heat for the first time.

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